When the Whole Vibrates Faster than Any of its Parts

Computational Superoscillation Theory

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Introduction. The phenomenon/theory of "superoscillations"—brought to my attention by my friend Ahmed Sebbar[1]—originated in the time-symmetric formulation of quantum mechanics[2] that gave rise to the theory of "weak measurements." Recognition of the role that superoscillations play in that work was brought into explicit focus by Aharonov *et al* in 1990, and since that date the concept has found application to a remarkable variety of physical subject areas.

Yakir Aharonov took his PhD in 1960 from the University of Bristol, where he worked with David Bohm.[3] In 1992 a conference was convened at Bristol to celebrate Aharonov's 60$^{\text{th}}$ birthday. It was on that occasion that the superoscillation phenomenon came first to the attention of Michael Berry (of the Bristol faculty), who in an essay "Faster than Fourier" contributed to the proceedings of that conference[4] wrote that "He [Aharonov] told me that it is possible for functions to oscillate faster than any of their Fourier components. This seemed unbelievable, even paradoxical; I had heard nothing like it before. . . " Berry was inspired to write a series of papers relating to the theory of superoscillation and its potential applications, as also by now have a great many other authors.

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[1] Private communication, 17 November 2017.

[2] Y. Aharonov, P. G. Bergmann & J. L. Libowitz, "Time symmetry in the quantum meassurement process," Phys. Rev. B **134**, 1410–1416 (1964).

[3] It was, I suspect, at the invitation of E. P Gross—who had collaborated with Bohm when both were at Princeton—that Aharonov spent 1960–1961 at Brandeis University, from which I had taken the first PhD and departed to Utrecht/CERN in February of 1960, so we never met. Aharonov is presently attached jointly to Tel Aviv University, the Perimeter Institute and Chapman University, where Ahmed Sebbar—formerly at the University of Bordeaux—has recently joined the mathematics faculty, which accounts for Ahmed's recent interest in superoscillations.

[4] *Quantum Coherence and Reality: In Celebation of the 60th Birthday of Yakir Aharomov* (1994), available on the web.

My objective here will be to provide a *Mathematica*-based survey of the most basic essentials of the superoscillatory phenomenon. I work principally from a paper "The mathematics of superoscillations" by Y. Aharonov *et al*[5]. Chapter 2 of that paper provides a review of the quantum mechanical origins of the subject. Superoscillations make their entrance in §2.5 ("Large weak values and superoscillations," page 22). I pick up the story in Chapter 3 ("Basic mathematical properties of superoscillating sequences," page 33).

Preparatory trivialities, and some terminology. Familiarly, the Fourier series

$$S(x) = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\sin(mx)$$

—the limit of the partial sums

$$S_n(x) = \frac{2}{\pi}\sum_{m=1}^{n}\frac{(-1)^{m+1}}{m}\sin(mx) \quad : \quad n\to\infty$$

—provide a representation of the "sawtooth function", to which the functions $S_n(x)$ provide progressively better approximations (Figure 16). Aharonov calls functions of the form $S_n(x)$—and more generally of the form

$$\sum_{m=0}^{n} a_m \begin{cases}\exp(ik_m x), & \text{else}\\ \cos(ik_m x), & \text{else}\\ \sin(ik_m x)\end{cases} : \begin{cases}\text{weighted sums of finitely many}\\ \text{oscillatory Fourier components}\end{cases}$$

—"Fourier sequences." In this language, Fourier series are limits of Fourier sequences.

Look to the function

$$F_n(x) \equiv \big[\cos\big(\tfrac{x}{n}\big) + i\sin\big(\tfrac{x}{n}\big)\big]^n = \big[\exp\big(i\,\tfrac{x}{n}\big)\big]^n = e^{ix} = \cos x + i\sin x$$

which is evidently the simplest possible Fourier sequence/series. Binomial expansion gives

$$F_n(x) = \sum_{p=0}^{n} i^p \tbinom{n}{p}\cos^{n-p}(x) sin^p(x)$$

the real/imaginary parts of which provide an infinite set of identities, of which the leading instances

$$\begin{aligned}\cos x &= \cos^2(\tfrac{1}{2}x) - \sin^2(\tfrac{1}{2}x)\\ \sin x &= 2\cos(\tfrac{1}{2}x)\sin(\tfrac{1}{2}x)\end{aligned}$$

—usually written

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ \sin 2x &= 2\cos x\sin x\end{aligned}$$

[5] Y. Aharonov, F. Colombo, I. Sabadini, D. C. Struppa & J. Tollaksen. The paper is available on the web as an arXiv preprint (5 November 2015) and as a Memoir of the American Mathematical Society (2017).

—are the familiar "double angle formulæ." Similar arguments lead directly to multiple angle formulæ of ascending order; thus

$$\begin{aligned}\cos 3x &= \cos^3 x - 3\cos x\sin^2 x\\ \sin 3x &= 3\cos^2 x\sin x - \sin^3 x\end{aligned}$$

$$\begin{aligned}\cos 4x &= \cos^4 x - 6\cos^2 x\sin^2 x + \sin^4 x\\ \sin 4x &= 4\cos^3 x\sin x - 4\cos x\sin^3 x\end{aligned}$$

The equivalence of the expressions on the left/right sides of such equations can be established in *Mathematica* by means of the `TrigExpand` and `TrigReduce` commands; thus

$$\begin{aligned}\texttt{TrigExpand}[\cos 2x] &= \cos^2 x - \sin^2 x\\ \texttt{TrigReduce}[\cos^2 x - \sin^2 x] &= \cos 2x\end{aligned}$$

All of which hinges on the elementary fact that $e^{inx} = (e^{ix})^n$.

By slight adjustment—vistas of a new superoscillatory world. Look now, with Aharonov, to the functions

$$F_n(x,a) \equiv \left[\cos\left(\tfrac{x}{n}\right) + i\,a\sin\left(\tfrac{x}{n}\right)\right]^n \tag{1.1}$$

which give back the functions $F_n(x) = e^{ix}$ (all n) in the case $a = 1$. Writing

$$\begin{aligned}&= \left[\frac{e^{ix/n}+e^{-ix/n}}{2} + i\,a\,\frac{e^{ix/n}-e^{-ix/n}}{2i}\right]^n\\ &= \left[\frac{1+a}{2}\,e^{ix/n} + \frac{1-a}{2}\,e^{-ix/n}\right]^n \qquad (1.2)\\ &= \sum_{p=0}^{n} \tfrac{1}{2^n}\tbinom{n}{p}(1+a)^{n-p}(1-a)^p e^{ix(n-p)/n}e^{-ixp/n}\\ &= \sum_{p=0}^{n} B_n(p,a)\,e^{ik_n(p)x} \qquad (2)\end{aligned}$$

$$\begin{aligned}B_n(p,a) &= \tfrac{1}{2^n}\tbinom{n}{p}(1+a)^{n-p}(1-a)^p\\ k_n(p) &= 1 - 2p/n\end{aligned}$$

we see from (2) that each $F_n(x,a)$ is a Fourier sequence, a weighted sum of $n+1$ Foruier terms $e^{ik_n x}$. To distinguish such sequences from Fourier sequences in general I will call them "Aharonov sequences."

The wave numbers $k_n(p) : p = 0, 1, \ldots, n$ proceed in n equal steps from $k_n(0) = 1$ to $k_n(n) = -1$, and in all cases $|k_n(p)| \leqslant 1$.[6] The associated wave lengths $\lambda_n(p) = 2\pi/|k_n(p)|$ are therefore *all greater than* 2π.

[6] Note that $k_n(\frac{n}{2}) = 0$, so $F_n(x,a)$ includes a constant term iff n is even.

Working either from (1.1) or (1.2), we find that

$$F_n(x,a) \approx [1 + iax/n]^n \quad : \quad \frac{x}{n} \ll 1$$

so for any given x

$$\lim_{n\to\infty} F_n(x,a) = e^{iax} \tag{3}$$

which at $a = 1$ gives back the result discussed in the preceeding section, but for $a > 1$ *has wave number greater, and wavelength shorter, than that of any of the contributory Fourier terms*. Taking our language not from the space domain but from the time domain, we have in the limit $n \to \infty$ an angular frequency greater, and period shorter, than that of any of the contributory Fourier terms, whence my title: *the whole vibrates faster than any of its parts*. That, in a nutshell, is the essence of the superoscillatory phenomenon.

Graphic evidence. Generally, one expects properties that hold in a limit to be approximated ever better, to become ever more vividly evident as one *approaches* the limit; that is the principle that—inevitably, since ∞ lies forever out of reach—informs the following graphic experiments/demonstrations.

The functions $F_n(x,a)$, e^{iax}, etc. are complex-valued. For purposes of graphic display one must look to their real (else imaginary) parts or (for some purposes more usefully) to their absolute values or (for most purposes much less usefully) to their phases.

THE CASE $F_{20}(x,2)$

Figures 1 & 2 show respectively the $\cos(x/n)$ and $\sin(x/n)$ that enter into the construction of $F_{20}(x,2)$. All have wavelengths $\geqslant 2\pi$. The real parts of $F_{20}(x,2)$ and e^{i2x} are superimposed in Figure 3a, their imaginary parts in Figure 4a. Even though the asymptote e^{i2x} has wavelength $\lambda = \pi < 2\pi$, the coincidence is seen to be reasonably good on the interval $|x| < \frac{1}{2}\pi$. Figure 3b shows the absolute value of the difference between the real parts of $F_{20}(x,2)$ and e^{i2x}, while Figure 4b shows the does the same for their imaginary parts; those figures show in particular that the error grows rapidly as one moves away from $x = 0$. Figure 5 (the same as Figure 3b, but with extended range) shows that the error grows to an enormous (but finite) size, but is *periodic*, with period $\xi_{20} = 20\pi$.[7]

EFFECT OF INCREASING n, WITH a HELD CONSTANT

Figure 6 differs from Figure 3a only in that the value of n has been increased from 20 to $20^2 = 400$. In Figure 7 (note the extended range) the value of n has been further increased to $20^3 = 8000$. The upper bound of the "domain of good approximation" has increased from about $\frac{1}{2}\pi$ to about $2\pi = 2^2(\frac{1}{2}\pi)$ to about $8\pi = 2^2(2\pi) = 2^2 2^2(\frac{1}{2}\pi)$. We might, on that informal basis, expect in the case $n = 20^4 = 160000$ to find that the domain of good approximation extends to about $x = 2^2 2^2 2^2(\frac{1}{2}\pi) = 32\pi$, which is an expectation supported by Figure 8.

[7] $\exp(x/n)$ has period $2\pi n$, so $|\exp(x/n)|$ has period $\xi_n = \pi n$.

EFFECT OF INCREASING a, WITH n HELD CONSTANT

Figure 9 differs from Figure 7 in only one respect: the parameter a has been doubled (increased from 2 to 4) and the upper bound of the domain of good approximation has been reduced from from about 8π to about $2\pi = (\frac{1}{2})^2 8\pi$. A second doubling of the parameter ($2 \to 4 \to 8$) produces Figure 10; here the location of the upper bound is difficult to estimate, but appears to lie in the neighborhood of $\frac{1}{2}\pi = (\frac{1}{2})^2(\frac{1}{2})^2 8\pi$. It is, in any event clear, (see Figures 11 and 12) that increasing the value of a decreases the width of the domain of good approximation, for which one can compensate by increasing the value of n.

QUALITATIVE EFFECTS OF $\{a, n\}$-VARIATION

The curves that connect maximal points in Figures 10, 11 & 12 were produced by commands of the form `Abs[`$F_n(x,a)$`]`.[8] Such curves can be used to eliminate a lot of irrelevant clutter and expose more clearly what is going on. Figure 13 shows the absolute values of $F_n(x,2) : n = \{10, 11, 12\}$. The periodicity noted previously is clearly evident. Gridlines at $\{\frac{1}{2}n\pi\}$ and $\{\frac{1}{2}n\pi + n\pi\}$ locate the maxima, which are found to have the values 1024, 2048 and 4096, respectively, which grow—each to the next—by factors of 2. Increasing the value of a from 2 to $3 = (\frac{3}{2})2$ produces Figure 14, in which the locations of the maxima remain unchanged, but their values have increased enormously, to 59049, 177147 and 531441, which grow by factors of 3. The pattern persists when we increase the value of a from 3 to $4 = (\frac{4}{3})3$, producing Figure 15; the locations of the maxima again remain unchanged, but their values have again increased enormously, to 1048576, 4194304 and 16777216, which grow by factors of 4.

Superoscillatory waveforms & associated graphics. Proceeding formally from

$$f(x) = \int g(k)e^{ikx}dk = \int g(k)\Big\{\lim_{n\to\infty} F_n(x,k)\Big\}dk$$
$$= \lim_{n\to\infty} f_n(x) \;:\; f_n(x) \equiv \int g(k)F_n(x,k)dk$$

we might—if $g(k)$ vanishes outside a bounded interval, and n is large enough to overwhelm the domain-contracting effect of k_{max}—expect $f_n(x)$ to provide a good and ever better approximation to $f(x)$. More particularly, if $f(x)$ refers to a real-valued periodic waveform (period 2π) we have

$$f(x) = \begin{cases} \sum_{m=0}^{\infty} g_m \cos(mx) & : \quad f(x) \text{ even} \\ \sum_{m=1}^{\infty} g_m \sin(mx) & : \quad f(x) \text{ odd} \end{cases} \tag{4}$$

From the Aharonov sequence $F_n(x,k)$ we construct

$$C_n(x,k) = \frac{F_n(x,k) + F_n(x,-k)}{2}$$
$$S_n(x,k) = \frac{F_n(x,k) - F_n(x,-k)}{2i}$$

[8] `Abs[`$x + iy$`]` $= \sqrt{x^2 + y^2}$.

and observe that

$$\lim_{n\to\infty} C_n(x,k) = \cos(kx) \quad : \quad \lim_{n\to\infty} S_n(x,k) = \sin(kx)$$

Suppose the Fourier coefficients g_m in (4) diminish so fast that truncation at $m = M$ produces what we can agree is a "good approximation"

$$f_M(x) = \sum_{m=0,1}^{M} g_m \begin{matrix}\cos(mx)\\ \sin(mx)\end{matrix}$$

to $f(x)$. We then expect

$$f_{M,n}(x) = \sum_{m=0,1}^{M} g_m \begin{matrix}C_n(x,m)\\ S_n(x,m)\end{matrix}$$

to produce, for n sufficiently large, a good "superoscillatory approximation" to $f(x)$. I look to examples:

It was remarked on page 2 that the sawtooth function $S(x)$—which is real, odd, has period 2π—results from setting

$$g_m = (-)^{m+1}\frac{2}{m\pi}$$

in the sine series. Truncation at $M = 15$ produces the function $S_{15}(x)$ displayed in Figure 16 (note the Gibbs phenomenon that results from the discontinuity of the sawtooth function). That same central portion of the superoscillatory approximation $S_{15,100000}(x)$ to $S(x)$ is shown in Figure 17. The figures are indistinguishable, which is remarkable because the Fourier components that enter into the construction of $S_{15}(x)$ all have wavenumbers $\geqslant 1$ (therefore wavelengths $\leqslant 2\pi$) while those that enter into the construction of $S_{15,100000}(x)$ all have wavenumbers $\leqslant 1$, therefore wavelengths *greater* than the period 2π of the figure. There is, however, a heavy price to be paid for this superoscillatory accomplishment, since

$$S_{15,100000}(100000\,\pi/2) \approx 3.52988 \times 10^{115438}$$

It would, in the absence of some heroically effective compression algorithm, require an unreasonably enormous amount of energy to achieve by any means a physical realization of such a wave.

We look to a second example. In

$$T(x) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\sin(mx) = \begin{cases} +1 & : \quad 0 < x < \pi \\ -1 & : \quad \pi < x < 2\pi \end{cases}$$

we have the Fourier representation of a square wave of period 2π. Truncation at $m = 15$ produces the function $T_{15}(x)$ plotted in Figure 18 (Gibbs' phenomenon is again conspicuous), of which the superoscillatory approximation $T_{15,100000}(x)$ is shown in Figure 19. Again, the two are indistinguishable *on the interval shown*, but differ radically (see again Figures 13–15) as x advances toward $100000\pi/2$. On the other hand, we expect $T_{15}(x)$ and $T_{15,n}(x)$ to come into precise agreement *on an unrestricted x-domain* in the (physically/computationally) unattainable limit $n \to \infty$.

Concluding comments. In my attempt to display as simply as possible the most essential features of the superoscillation phenomenon I have ridden roughshod over any number of mathematical niceties (for example: the distinction between "convergence" and "uniform convergence"). Those are treated in elaborate detail by Aharonov and colleagues in the paper cited previously.[5] Berry[4] used the method of stationary phase (saddlepoint method) to study some aspects of the mathematical theory.

So far as concerns applications, the subject was, as previously remarked, *born* of an application (theory of weak measurements), and Aharonov *et al*[5] look also to some of its other quantum ramifications; they look, for example, to the dynamical evolution of superoscillatory initial states under action of the Schrödinger equation (free particle, oscillator), the wave equation and other dynamical equations. Their bibliography, though most of the 132 papers cited refer to aspects of weak measurement theory, includes quite a number that allude in their titles to superoscillations. Michael Berry, in the §5. "Concluding Remarks" of the paper that announced his entry into the field,[4] remarks that "Aharonov's discovery. . . could have applications in several branches of physics," and proceeds to list some: "Perhaps more interesting are the superoscillatory functions of two variables, representing images. One envisages new forms of microscopy, in which structures much smaller than the the wavelength λ would be resolved by representing them as superoscillations." And in subsequent publications (cited by Aharonov) Berry proceeeded to explore some of those potential applications.[9]

I quote now from Berry's §4. "Beethoven at 1 Hz": "Professor I. Daubechies has informed me that superoscillations are known in signal processing, in the context of oversampling. This is a function faster than the Nyquist rate; *i.e.*, at points $x = n\pi$ where the function is bandlimited by $|k| \leqslant 1$. If a function is oversampled in a finite range, extrapolation outside this range is exponentially unstable.[10] She quotes B. Logan as saying that it is possible in principle to design a bandlimited signal with a bandwidth of 1 Hz that would reproduce Beethoven's 9$^{\text{th}}$ symphony exactly. With the superoscillatory

[9] "Evolution of quantum superoscillations, and optical superresolution without evanescent waves," (2006); "Superoscillation in speckle patterns," (2009); "Exact nonparaxial transmission of subwavelength detail using superoscillations," (2013)

[10] Y. Aharanov, J. Anandan, S. Popescu & L. Vaidman, Phys. Rev. Letters **64**, 2965 (1990).

functions described in this paper it is possible to give an explicit recipe for constructing this signal." Which Berry proceeds to do.[11] The Beethoven story occurs again (and is in some respects subverted) in A. Kempf, "Blackholes, bandwidths & Beethoven,"[12] which borrows much of its formalism from Berry. I have been able to discover nothing about the "B. Logan" responsible for this bit of dramatic frivolity.

[11] Ingrid Daubechies is a Belgian physicist/mathematician best known for her contribution to the theory of wavelets, and at the beginning of her career was a student/collaborator of Alex Grossmann when he was working as co-inventor of that subject; she was for many years at Princeton, is presently at Duke. Grossmann was a gloomy Romanian-French Harvard graduate student of Roy Glauber, who hung around Brandeis in the late 1950s because he and Glauber could not get along, so became my friend. Also victimized by Glauber was my Brandeis friend Evelyn Fox (Keller); she had been assured that "women are ill-equipped to study the foundations of quantum theory" by Glauber, who was a no-show at her qualifying exam, mentioned later (without apology) that he had slept late that day.

[12] arXiv:gr-qc/9907084v2 (3 Nov 1999).